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# AN ASYMPTOTIC SOLUTION OF A CLASS OF COUPLED EQUATIONS*

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We consider a class of coupled equations that is a generalization of a class studied earlier /1/. Such equations appear, in particular, when solving composite problems in elasticity theory for non-uniform bodies /2-5/. We define equations for which the approximate solutions, constructed using the method described in /1/ by reducing the problem to a finite system of algebraic equations, is two-sided asymptotically exact in terms of the characteristic geometric parameter of the problem. As an example we consider integral equations generated by Fourier and Hankel transformations.

1. Suppose we are given an integral transformation

$$g(x)=\int_a^b G(\gamma)B(\gamma,x)\,d\gamma,\quad G(\gamma)=\int_\alpha^\beta g(\xi)M(\gamma,\xi)\,d\xi \tag{1.1}$$

or an expansion

$$g(x)=\sum_{k=0}^{\infty} G_k B(\gamma_k,x),\quad G_k=\int_\alpha^\beta g(\xi)M(\gamma_k,\xi)\,d\xi \tag{1.2}$$

and the function $B(\gamma,x)$ is a solution of a linear differential equation of second order in $x$:

$$(L-\gamma^2)B(\gamma,x)=0,\quad L_\gamma B=r(x)[s(x)B']'+t(x)B\quad (a\leqslant x\leqslant b)\,. \tag{1.3}$$

Here $s(x)>0$ for $x\in(a,b)$ and the function $r(x)$ has constant sign for $x\in(a,b)$.

Suppose also that the functions $B$ and $B'$ are bounded as $x\to b$, and that at $x=a$ we have $\alpha_1B+\alpha_2B'=0$. Furthermore, the numbers $\gamma_k$ constitute a denumerable set of roots of some transcendental equation with $a\leqslant\gamma_k<\gamma_{k+1}\leqslant b$.

We assume that Eq.(1.3) satisfies the conditions of Fuchs's theorem /6/, i.e. the coefficient $p_i(x)$ of $d^{2-i}y/dx^{2-i}$ has the form $(x-\alpha)^{-i}P_i(x-\alpha)$, where the function $P_i(x-\alpha)$ is holomorphic in the domain of the point $\alpha$; (the condition of Fuchs's theorem is necessary and sufficient for Eq.(1.3) to have two independent integrals that are regular in the domain of the point $\alpha$).

We consider a coupled integral equation (a coupled series equation)

$$\int_a^b Q(\gamma)\rho(\gamma)K(\gamma\lambda)B(\gamma,x)\,dh(\gamma)=f(x),\quad c\leqslant x\leqslant d \tag{1.4}$$
$$\int_a^b Q(\gamma)B(\gamma,x)\,dh(\gamma)=0,\quad \alpha\leqslant x<c,\ d<x\leqslant\beta$$

where for (1.1) the function $h(\gamma)\equiv\gamma$, while for (1.2)

$$h(\gamma)\equiv\frac{1}{2}\sum_{k=0}^{\infty}[1+\operatorname{sgn}(\gamma-\gamma_k)]\,. \tag{1.5}$$

Here the function $\rho(\gamma)$ is such that for $K(\lambda\gamma)=1$ we know the solution of Eq.(1.4). Suppose /2/

$$K(\gamma)=A+B\gamma+v(\gamma^2),\quad \gamma\to 0;\quad K(\gamma)=1+D\gamma^{-1}+v(\gamma^{-2}),\quad \gamma\to\infty\,. \tag{1.6}$$

*Definition 1.1.* The function $K(\gamma)$ belongs to one of the classes $\Pi_N$ ($\Sigma_M$ and $S_{N,M}$) which have the forms

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$$\prod_N : K(\gamma\alpha) = K_N(\lambda\alpha) \equiv \prod_{i=1}^{N} \frac{\alpha^2 + A_i^2\lambda^{-2}}{\alpha^2 + B_i^2\lambda^{-2}}, \tag{1.7}$$

$$\sum_M : K(\lambda\alpha) = K_M^{\Sigma}(\lambda\alpha) \equiv \sum_{k=1}^{M} \frac{c_k\lambda^{-1}|\alpha|}{\alpha^2 + D_k^2\lambda^{-2}}, \tag{1.8}$$

$$S_{N,M} : K(\lambda\alpha) = K_N(\lambda\alpha) + K_M^{\Sigma}(\lambda\alpha). \tag{1.9}$$

Here $A_i, B_i$ $(i = 1, 2, \ldots, N)$, $C_k, D_k$ $(k = 1, \ldots, M)$ are certain constants.

We have Theorem 1.1 /3/: given the condition that the function $K(\gamma)$ satisfies the properties (1.6), it can be approximated by expressions of the form

$$K(\lambda\gamma) = K_N(\lambda\gamma) + K_\infty^{\Sigma}(\lambda\gamma). \tag{1.10}$$

In accordance with (1.1),

$$Q(\gamma) = \int_c^d q(\xi) N(\gamma, \xi) d\xi, \quad N(\gamma, \xi) = M(\gamma, \xi) \text{ for } \rho(\gamma) = 1. \tag{1.11}$$

Substituting (1.11) into (1.4), we obtain

$$\int_a^l \int_c^d q(\xi) \rho(\gamma) K(\lambda\gamma) N(\gamma, \xi) B(\gamma, x) d\xi dh(\gamma) = f(x), \quad c \leqslant x \leqslant d. \tag{1.12}$$

Below, the integral operator corresponding to the function $K(\gamma)$ belonging to the class $X$ will also be denoted by $X$.

Using (1.10), we rewrite (1.12) in operator form

$$\Pi_N q + \Sigma_\infty q = f. \tag{1.13}$$

In (1.13) the operator $\Pi_N$ corresponds in (1.10) to the function $K(\gamma)$ of form (1.7), and $\Sigma_\infty$ to the function $K(\gamma)$ of the form (1.8).

*Definition 1.2.* We shall say that Eq.(1.4) satisfies condition $A$ if for $K(\gamma) \in \Pi_N$ one can construct a closed solution, following /1/. We shall denote this solution by

$$q = \Pi_N^{-1} f, \quad x \in (c, d). \tag{1.14}$$

In other words, condition $A$ means that for the function $f(x)$, belonging to some class $W(c, d)$, there exists a function $q(x)$, belonging to some class $V(c, d)$, such that the equality (1.4) is true.

From the representation (1.14) it follows that

$$\| q \|_{V(c,d)} \leqslant m(\Pi_N) \| f \|_{W(c,d)}, \quad m(\Pi_N) = \text{const}.$$

Below we shall use $m(X)$ to denote some constant that depends on the specific form of the function $X$.

2. On the basis of the Hahn-Banach theorem /7/ we shall show that, if certain conditions are satisfied, expression (1.14) is an asymptotically exact solution of Eq.(1.13) as $\lambda \to 0$ and $\lambda \to \infty$.

As a preliminary we consider the question of the existance and uniqueness of the solution of the coupled Eqs.(1.4) for functions $K(\gamma)$ belonging to class $S_{N,M}$; in this case it can be written in the form

$$\Pi_N q + \Sigma_M q = f. \tag{2.1}$$

We shall determine the conditions under which the operator $\Pi_N^{-1}\Sigma_M$ of Eq.(1.4) is a contraction operator /7/. For this we use the following assertion.

*Lemma 2.1.* If $\gamma\rho(\gamma) = r^{-1}(\gamma)$, $M(\gamma, x) = B(\gamma, x)$, and $a$ is a real number, then the bilinear form

$$\alpha_{ia}(\xi, x) \equiv \int_a^b \frac{\gamma\rho(\gamma) M(\gamma, \xi) B(\gamma, x)}{\gamma^2 + a^2} dh(\gamma)$$

can be written in the form

$$\alpha_{ia}(\xi, x) = \begin{cases} B_-(ia, \xi) B_+(ia, x), & \xi < x \\ B_+(ia, \xi) B_-(ia, x), & x < \xi \end{cases}$$

where $B_-(ia, x)$ and $B_+(ia, x)$ are linearly independent solutions of Eq.(1.3) such that $B_-(ia, \xi) \to 0$ and $B_+(ia, \xi) \to \infty$ as $a \to \infty$.

The assertion of Lemma 2.1 is proved by putting $\gamma_r = ia$ in Lemma 28.1 in reference /8/. Without loss of generality we put $M = 1$ in (2.1).

*Lemma 2.2.* If Eq.(1.4) satisfies condition $A$ and the conditons of Lemma 2.1, then the operator $\Sigma_1 q$ in (2.1) can be represented in the form of a series /9/, ($\Sigma_1 q$ corresponding to $K_1^\Sigma(\lambda\gamma)$):

$$\Sigma_1 q = \sum_{k=0}^{\infty} \beta_k B(\gamma_k, x), \quad \beta_k(a) = \frac{c_r \lambda^{-1}}{\gamma_k^2 - a^2} \left[ C(a) \int_c^d q(\xi) B(\gamma_k, \xi)\, d\xi - \right. \tag{2.2}$$

$$\left. s(c)\, W_c^a(B_+, B)\, I_- + s(d)\, W_d^a(B_-, B)\, I_+ \right],$$

$$I_\pm = \int_c^d q(\xi) B_\pm(a, \xi)\, d\xi, \quad a = i\frac{D}{\lambda},$$

$$W_b^a(A, B) = A(a, b) B'(\gamma_k, b) - B(\gamma_k, b) A'(a, b).$$

Here $\gamma_0, \gamma_1, \ldots, \gamma_n \ldots$ is the set of all eigenvalues of problem (1.3) with associated boundary conditions, $B(\gamma_k, x)$ are the corresponding normalized eigenfunctions, and $C(a)$ is a bounded constant, fixed for each equation in (1.3), connected with the Wronskian determinant $W(B_+, B_-$ of the functions $B_+(a, x)$ and $B_-(a, x)$ by the relation

$$W[B_+(a, x), B_-(a, x)] = C(a)\, s^{-1}(x).$$

To prove Lemma 2.2 we write down the representation of the expansion coefficients $\beta_k$:

$$\beta_k(a) = \frac{c_r}{\lambda} \int_c^d q(\xi) A_k(a, \xi)\, d\xi, \quad A_k(\alpha, \xi) = \int_c^d \alpha_a(\xi, x) \frac{B(\gamma_k, x)}{r(x)}\, dx. \tag{2.3}$$

Using Lemma (2.1) and a well-known property of solutions of second-order differential equations /10/

$$\int_c^d \frac{B(a, x) B(ib, x)}{r(x)}\, dx = \left[ \frac{s(x)}{a^2 + b^2} \left( B'(a, x) B(ib, x) - B(a, x) B'(ib, x) \right) \right]_c^d$$

where $B(a, x)$ and $B(ib, x)$ are any two solutions of Eq.(1.3) corresponding to $\gamma = a$ and $\gamma = ib$, the second expression (2.3) can be rewritten in the form

$$A_k(a, \xi) = \frac{1}{\gamma_k^2 - a^2} \begin{cases} s(x) B_-(a, \xi) [B_+(a, x) B'(\gamma_k, x) - B(\gamma_k, x) B_+'(a, x)] \,|_c^\xi, & \xi < x \\ s(x) B_+(a, \xi) [B_-(a, x) B'(\gamma_k, x) - B(\gamma_k, x) B_-'(a, x)] \,|_\xi^d, & x < \xi \end{cases}$$

from which the assertion of Lemma 2.2 follows.

3. We consider Eq.(1.3) and put $y(x) = B(x)\sqrt{s(x)}$. We obtain the equation

$$y'' - \gamma^2 q(x)\, y = 0; \quad q(x) = p(x) - R(x)\gamma^{-2} \tag{3.1}$$

$$p(x) = (rs)^{-1}, \quad R(x) = t(rs)^{-1} - s''(2s)' + {}^1/_4 (s's^{-1})^2$$

for $y(x)$.

*Lemma 3.1.* When the conditions of Lemma 2.2 are satisfied the operator $\Pi_N^{-1}\Sigma_M$ of Eq. (1.4) is a contraction operator in the space $V(c, d)$ if 1) the function $q''(x)$ is continuous for $x \in (a, b)$, and 2) $q(x) \geqslant 0$ for $x \in (a, b)$ when $0 < \lambda < \lambda^*$ where $\lambda^*$ is some fixed value for $\lambda$.

To complete the proof we will estimate the coefficients $\beta_k$ in (2.2) in terms of $\lambda$. We use the notation

$$F_\pm(a, e) = B_\mp(a, \xi)\, W_e^a(B_\pm, B) \quad (e = c, d).$$

According to Theorem 2 from /11/, when conditions 1 and 2 are satisfied, Eq.(3.1) has a solution of the form

$$y_{1,2}(x, \gamma) = q^{-1/4}(x)\, E_\pm(x_0, x)\, [1 + \gamma^{-1} \varepsilon_{1,2}(x, \gamma)], \tag{3.2}$$

$$E_\pm(x_0, x) = \exp\left\{ \pm\gamma \int_{x_0}^x \sqrt{q(t)}\, dt \right\}.$$

The estimates

$$|\varepsilon_j(x,\gamma)| \leqslant c, \quad x \in [a,b], \quad \gamma \geqslant \gamma_0 > 0, \quad j = 1, 2 \tag{3.3}$$

hold for the functions $\varepsilon_{1,2}$ where the constant $c$ is independent of $x$ and $\gamma$.
The asymptotic form of (3.2) can be differentiated, i.e.

$$y_{1,2}'(x,\gamma) = \pm\gamma q^{1/4}(x) E_{\pm}(x_0, x)\left[1 + \gamma^{-1}\varepsilon_{1,2}(x,\gamma)\right], \tag{3.4}$$

where estimates of the form (3.3) hold for the functions $\varepsilon_{1,2}$. Using (3.2) and (3.4), and taking into account that $\gamma = D_1\lambda^{-1}$ in (3.1), we deduce the following: because $c < \xi$, the behaviour of the function $F_+(\gamma, c)$ is determined by a multiplier of the form $E_-(c,\xi)$ and one can find a $\gamma_0$ such that $F_+(\gamma, c) \to 0$ for $\gamma \geqslant \gamma_0 > 0$.

As above, because $d > \xi$, the existence of the multiplier $E_-(\xi, d)$ enables one to find a $\gamma_0$ such that $F_-(\gamma, d) \to 0$ for $\gamma \geqslant \gamma_0 > 0$.

Thus for $0 < \lambda < \lambda_1$ where $(\lambda_1 = D_1\gamma_0^{-1})$, using the fact that the expansion coefficients of (2.2) have the form (2.3) and that the functions $B(\gamma_k, x)$ are orthonormalized, we obtain the estimate

$$\|\Sigma_1 q\|_{V(c,d)} \leqslant \sum_{k=0}^{\infty} |a_k| \leqslant \lambda M^*, \quad \lambda \to 0 \ (0 < \lambda < \lambda_1)$$

where the constant $M^*$ does not depend on $\lambda$. From this it follows that $\lambda$ can be chosen so that the operator $\Pi_N^{-1}\Sigma_M$ is a contraction operator /7/ under the conditions of the present lemma.

**4.** We shall investigate the conditions under which the solution (1.14) is an asymptotically exact solution of Eq.(1.4) as $\lambda \to \infty$ $(\gamma \to 0)$. To do this, following the scheme described earlier, we determine the conditions under which the operator $\Pi_N^{-1}\Sigma_M$ of Eq.(1.4) is a contraction operator.

Throughout the following we shall assume that the solutions of Eq.(1.3) satisfy symmetry conditions:

$$B(\gamma, x) = B(x, \gamma). \tag{4.1}$$

In agreement with condition (4.1) the behaviour of $B(\gamma, x)$ as $\gamma \to 0$ is determined by the behaviour of the corresponding solution of Eq.(1.3) as $x \to 0$.

We reduce Eq.(1.3) to selfconjugate form by multiplying it by the function $r^{-1}(x)$. From (1.3) we obtain

$$L_\gamma B(\gamma, x) = [s(x)B']' - Q(x)B = 0, \quad s(x) > 0, \quad a \leqslant x \leqslant b \tag{4.2}$$
$$Q(x) = [t(x) - \gamma^2]\, r^{-1}(x).$$

We assume that the coefficients $s(x)$ and $Q(x)$ of Eq.(4.2) are analytic in the disk $|x| < R$. Then any solution $B(x)$ of Eq.(4.2) are analytic in this disk, i.e. can be expanded in a power series converging inside the disk $|x| < R$ /12/.

*Lemma 4.1.* The operator $\Pi_N^{-1}\Sigma_M$ of Eq.(1.4) is a contraction operator in the space $V(c,d)$ if the coefficients $s(x)$ and $Q(x)$ of Eq.(4.2) are analytic on the disk $|x| < R$ for $\lambda > \lambda^a$, where $\lambda^a$ is some fixed value of $\lambda$, and if the symmetry condition (4.1) is satisfied.

To prove the lemma we estimate the coefficients $\beta_k$ in (2.2) in terms of $\lambda$. From the conditions of the lemma and the symmetry condition (4.1) it follows that one can find a $\lambda^a$ such that for $\lambda > \lambda^a$ the solutions $B_{\pm}(a, x)$ can be represented in the form of power series in $\lambda^{-1}$, converging in the disk $|\lambda| > \lambda^a$. From this it follows that

$$\|\Sigma_1 q\|_{V(c,\ d)} \leqslant \sum_{k=0}^{\infty} a_k < \frac{M^a}{\lambda}, \quad \lambda \to \infty \ (\lambda > \lambda^a)$$

where the constant $M^a$ does not depend on $\lambda$.

Thus $\lambda$ can be chosen so that the operator $\Pi_N^{-1}\Sigma_M$ is a contraction operator /7/ under the conditions of the present lemma $(\lambda^a = M^a)$.

We shall consider separately the case of Eq.(4.2) when the point $x = 0$ is a regular singular point, i.e.

$$s(x) = x\varphi(x), \quad \varphi(0) \neq 0 \tag{4.3}$$

where $\varphi(x) > 0$ is a continuous function in $[a, b]$. We note that the function $s(x)$ of the form (4.3) satisfies the conditions of Fuchs's theorem.

The following lemma holds (/13/, p.628):

*Lemma 4.2.* Suppose $B_+(x)$ and $B_-(x)$ are two linearly independent solutions of Eq.(4.2), whose coefficient $s(x)$ satisfies condition (4.3). Then if $B_+(0) \neq 0$, $B_-(x)$ has a logarithmic singularity at $x = 0$. If $B_+(x)$ has an $n$-th order zero $(n > 0)$ at $x = 0$, then $B_-(x)$ has an $n$-th order pole at $x = 0$.

*Lemma 4.3.* Suppose that the coefficient $s(x)$ of Eq.(4.2) has the form (4.3), that condition (4.1) is satisfied and that

$$s(c)\,B(\gamma_k, c) = s(d)\,B(\gamma_k, d).$$

In this case the operator $\Pi_N^{-1}\Sigma_M$ of Eq.(1.4) is a contraction operator in the space $V(c, d)$ for $\lambda > \lambda^a$, where $\lambda^a$ is some fixed value of $\lambda$.

We shall estimate the coefficients $\beta_k$ in (2.2). From Lemma 4.2 and condition (4.1) it follows that there exists a $\lambda^a$ such that for $\lambda > \lambda^a$, if $B_+(0) \neq 0$, then

$$\|\Sigma_1 q\|_{V(c,\,d)} \leqslant \sum_{k=0}^{\infty} |a_k| \leqslant \frac{M_1 \ln \lambda}{\lambda}, \quad \lambda \to \infty \ (\lambda > \lambda^a) \tag{4.4}$$

and if $B_+(x)$ has an $n$-th order zero at $x = 0$, then in estimate (4.4) $M_1\lambda^{-1}\ln\lambda$ has to be replaced by $M_2\lambda^{-1}$.

Based on Lemmas 3.1, 4.1, and 4.3, applying the contraction mapping principle to the equation

$$q + \Pi_N^{-1}\Sigma_M q = \Pi_N^{-1} f$$

we obtain a proof of the existence and uniqueness of the solution of Eq.(2.1) under the conditions imposed.

Thus we have proved the following theorem.

*Theorem 4.1.* Eq.(1.4) is uniquely solvable in the space $V(c, d)$ for $K(\gamma)$ of class $S_{N,M}$ when the conditions of Lemmas, 3.1, 4.1 or 4.3 are satisfied, and the estimate

$$\|q(x)\|_{V(c,\,d)} \leqslant m(\Pi_N, \Sigma_M)\,\|f\|_{W(c,\,d)} \tag{4.5}$$

holds.

We have furthermore the following theorem.

*Theorem 4.2.* Eq.(1.4) is uniquely solvable in the space $V(c, d)$ for $K(\gamma)$ possessing properties (1.6) for $\gamma\rho(\gamma) = r^{-1}(\gamma)$ and satisfying condition $A$ and conditions 1) and 2) from Lemma 3.1 if $0 < \lambda < \lambda^*$, and also for $\lambda > \lambda^a$ when the conditions of Lemma 4.1 or 4.3 are satisfied ($\lambda^*$ and $\lambda^a$ being some fixed values of $\lambda$) and estimate (4.5) holds with $\Sigma_M$ replaced by $\Sigma_\infty$.

Theorem 4.2 follows from the assertions of Theorems 1.1 and 4.1 and is proved with the help of a trick used in perturbation theory, based on the method of successive approximations, as in /14/.

5. Examples of representations of the form (2.2).

1) $t(x) = 0,\ r(x) = s(x) = \text{const}$ in (1.3)

$$B(\alpha, \xi) = \cos\alpha\xi, \quad B_-(iD, \xi) = {}^1/_2\pi D^{-1}\exp(-D\xi)$$

$$B_+(iD, x) = \operatorname{ch} Dx$$

$$a_k\left(\frac{iD}{\lambda}\right) = \frac{4\pi c\lambda^{-1}}{(k\pi)^2 + D^2\lambda^{-2}}\left[\int_0^1 q(\xi)\left[\cos k\pi\xi - \exp\left(-\frac{D}{\lambda}\right)\cos k\pi \operatorname{ch}\frac{D}{\lambda}\xi\right]d\xi\right].$$

Here the conditions of Lemmas 3.1 and 4.1 are satisfied. The space $V(c, d) \equiv C_{1/2}^{(0)+}(-1, 1)$, where $C_{1/2}^{(0)+}(-1, 1)$ is the space of even functions that are continuous and have weight $(1 - x^2)^{1/2}$ with norm /14/

$$\|f\|_{C_{1/2}^{(0)+}(-1,1)} = \max_{x \in [-1,1]} f(x)(1 - x^2)^{1/2}.$$

$W(c, d)$ is the space of functions with first-order derivatives in the internal $[-1, 1]$ satisfying the Hölder condition with index $1/2 + \varepsilon$, with the usual norm /14/.

2) $r(x) = x^{-1},\ s(x) = x,\ t(x) = -n^2x^{-2}$ in (1.3)

$$B(\alpha, \xi) = J_n(\alpha\xi), \quad B_-(iD, \xi) = K_n(D\xi), \quad B_+(iD, x) = I_n(Dx) \quad (n = 0, 1).$$

Here $J_n(x)$ is Bessel function, and $I_n(x)$ and $K_n(x)$ are modified Bessel functions. We have

$$a_k\left(\frac{iD}{c}\right)=\frac{2c\lambda^{-1}}{J_{n+1}^2(\mu_k)(\mu_k^2+D^2\lambda^{-2})}\times$$
$$\left[\int_0^1 q(\rho)\rho\left[J_n(\mu_k\rho)-K_n\left(\frac{D}{\lambda}\right)\mu_k J_{1-n}(\mu_k)I_n\left(\frac{D}{\lambda}\rho\right)\right]d\rho\right]$$

$(n=0,\ 1)$, where for $n=0$ $V(c,d)\equiv C_{1/2}^{(0)+}(-1,\ 1)$, while for $n=1$ $V(c,d)\equiv C_{1/2}^{(0)-}(-1,1)$, where $C_{1/2}^{(0)-}$ is the space of odd functions, continuous with weight $(1-x^2)^{1/2}$. The corresponding space $W(c,d)$ is defined in /3/.

Here, for $n=0\ (n=1)$ the conditions of Lemma 3.1 are satisfied and estimate (4.4) holds (estimate (4.4) with $M_1\lambda^{-1}\ln\lambda$ replaced by $M_2\lambda^{-1}$).

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